Strong Coupling Limit of the Kardar-Parisi-Zhang Equation in $2+1$ Dimensions

F. Shahbazi [a], A. A. Masoudi [a], and M. Reza Rahimi Tabar [b,c]

[a] *Department of Physics, Sharif University of Technology, P.O.Box 11365-9161, Tehran, Iran.*

[b] *CNRS UMR 6529, Observatoire de la Côte d'Azur, BP 4229, 06304 Nice Cedex 4, France,*

[c] *Department of Physics, Iran University of Science and Technology, Narmak, Tehran 16844, Iran.*

Abstract

A master equation for the Kardar-Parisi-Zhang (KPZ) equation in 2+1 dimensions is developed. In the fully nonlinear regime we derive the finite time scale of the singularity formation in terms of the characteristics of forcing. The exact probability density function of the one point height field is obtained correspondingly.

PACS: 05.45.-a, 68.35.Ja, 02.40.Xx.

I. INTRODUCTION

Because of technical importance and fundamental interest, a great deal of efforts have been devoted to the understanding of the mechanism of thin-film growth and the kinetic roughening of growing surfaces in various growth techniques. Analytical and numerical treatments of simple growth models suggest that, quite generally, the height fluctuations have a self-similar character and their average correlations exhibit a dynamical scaling form. Numerous theoretical models have been proposed, of which the simplest nontrivial example is the Kardar-Parisi-Zhang equation [1]

$$\frac{\partial h}{\partial t} - \frac{\alpha}{2}(\nabla h)^2 = \nu \nabla^2 h + f(x, y, t), \tag{1}$$

where $h(\mathbf{x}, \mathbf{t})$ specifies the height of the surface at point $\mathbf{x}$ and f is a zero-mean, statistically homogeneous, white in time Gaussian process with covariance $\langle f(\mathbf{x}, t) f(\mathbf{x}', t')\rangle = 2D_0 \delta(t - t') D(\mathbf{x} - \mathbf{x}')$. Typically the spatial correlation of forcing is considered to be as a delta function, mimicking the short range correlation. Here we consider the spatial correlation as $D(\mathbf{x} - \mathbf{x}') = \frac{\mathbf{1}}{\pi \sigma^{\mathbf{2}}} \exp(-\frac{(\mathbf{x}-\mathbf{x}')^{\mathbf{2}}}{\sigma^{\mathbf{2}}})$, where σ is the variance of $D(\mathbf{x} - \mathbf{x}')$. When the variance σ is much less than the system size L , i.e. $\sigma \ll L$, the model represents a short range character for the forcing. This same equation is believed to describe the statistics of directed polymers in a random medium [8], where h has the meaning of the free energy. The average force on the interface is not essential and can be removed by a simple shift in h. Every term in the eq.(1) involves a specific physical phenomenon contributing to the surface evolution. The parameters ν, α and D_0 (and σ) are related to surface relaxation, lateral growth and the noise strength, respectively. The effective coupling constant for the KPZ equation is given by $g = \frac{2\alpha^2 D_0}{\nu^3}$. The limit $g \to \infty$ is known as the strong coupling limit. Despite the intense effort in recent years, the properties of the strong coupling phase are rather poorly understood. Only the critical exponents of the strong-coupling regime ($g \to \infty$ or $\nu \to 0$) are known in 1+1 dimensions and their values in higher dimensions as well as properties of the roughening transition have been known only numerically and by the various approximative schemes.

Theoretical richness of the KPZ model is partly due to its close relationships with other areas of statistical physics. It is shown that there is a mapping between the equilibrium statistical mechanics of a two dimensional smectic-A liquid crystal onto the non-equilibrium dynamics of the (1+1)- dimensional stochastic KPZ equation [3]. It has been shown [4] that, one can map the kinetics of the annihilation process $A + B \to 0$ with driven diffusion onto the (1+1)-dimensional KPZ equation. Also the KPZ equation is closely related to the dynamics of a Sine-Gordon chain [5], the driven-diffusion equations [6,7], and directed paths

in the random media [8] and charge density waves [9], dislocations in disordered solids [10], formation of large-scale structure in the universe [11,12], Burgers turbulence [13] and etc.

In this paper we attempt to clarify the short time physics of KPZ equation in the strong coupling limit by presenting an exact solution of the model in 2+1 dimensions. It turns out that eq.(1) exhibits a finite time singularity at time scale t_c. Using the master equation method it is shown that t_c can be expressed in terms of the characteristics of forcing as, $t_c = \frac{1}{4}(\frac{\pi}{8\alpha^2 D_0})^{1/3}\sigma^2$. Also we derive the time dependence of the moments $< (h-\bar{h})^n >$ for time scales before creation of singularities.

The main properties of the KPZ equation in the strong-coupling limit are as follow: (i) In the limit of $\nu \to 0$ the unforced KPZ (Burgers) equation develops singularities for the given dimension. In one spatial dimension it develops sharp valleys (point shocks). In the singular points the height gradients are not continuous [14]. In two spatial dimensions KPZ equation develops three types of singularities, the first singularities are finite sharp valley lines (shock lines) across which the height gradients are discontinuous. The second type is the end point of the sharp valley lines which separates the regular points and singular region and is called a kurtoparabolic point. As time increases these sharp valley lines hit each other and crossing point of two valley lines (shock lines) produces a valley node (shock node). Generically kurtoparabolic points disappear at large times and only a network of sharp valley lines survive [11]. A complete classification of the singularities of KPZ (Burgers) equation in two and three dimensions, by considering the metamorphoses of singularities as time elapses, has been done in [15]. (ii) For white in time and smooth in space forcing, the sharp valley lines (cusp lines) are smooth curves and the singularity structures of the forced case is similar to unforced problem. It is shown in [16] that in the stationary state the sharp valley lines produce a curvilinear hexagon lattice in two dimensions and, therefore, one finds a hexagonal tiling of singularity lines (valley lines).

Here we shall adapt the master equation approach which enables us to investigate the KPZ equation in the strong coupling limit [14]. It is one of the advantages of this method that allows us to write all the nonlinearities due to the nonlinear term $\frac{\alpha}{2}(\nabla h)^2$ in a closed form. When σ is finite, the nonlinear term develops *finite time* singularities for the height derivatives in the limit ($\nu \to 0$). Therefore, one would distinguish between different time regimes before and after the sharp valley formation. Starting from a flat initial condition, i.e. $h(x,0) = 0, h_x = u(x,0) = 0$ and $h_y = v(x,0) = 0$, we aim at tracing the time evolution given by the KPZ equation with $\nu \to 0$. Only for time scales before the sharp valley formation ($t < t_c$) the limit $\nu \to 0$ and $\nu = 0$ are equivalent. For $t > t_c$, the anomalous contributions of the diffusion terms to the equation of probability distribution function will break this equivalence [14]. Therefore, for $t < t_c$, one can impose $\nu = 0$ in the dynamics

and try to find realisable statistical solutions for the problem. We show that it is possible to find finite time realisable solutions for the probability density function of the height field at least in the early steps of the evolution of the surface. We attribute the finite life time of the realisable ensemble of statistical solutions to the fact that after a finite time the derivatives of the field $h(x,t)$ become singular and , therefore, the diffusion term contributions are no more negligible.

II. CALCULATION OF THE HEIGHT MOMENTS BEFORE THE SINGULARITY FORMATION

Let us consider the Kardar-Parisi-Zhang equation in 2+1 dimensions,

$$h_t(x,y,t) - \frac{\alpha}{2}(h_x^2 + h_y^2) = \nu(h_{xx} + h_{yy}) + f(x,y,t). \tag{2}$$

Defining $h_x(x,y,t) = u(x,y,t)$ and $h_y(x,y,t) = v(x,y,t)$, differentiating the above equation with respect to x and y and neglecting the viscosity term in the limit of $\nu \to 0$ before the creation of singularities we get

$$u_t(x,y,t) = \alpha(u(x,y,t)u_x(x,y,t) + v(x,y,t)u_y(x,y,t)) + f_x(x,y,t) \tag{3}$$

$$v_t(x,y,t) = \alpha(u(x,y,t)v_x(x,y,t) + v(x,y,t)v_y(x,y,t)) + f_y(x,y,t). \tag{4}$$

Now introducing Θ as

$$\Theta = \exp\Big(-i\lambda(h(x,y,t) - \bar{h}(t)) - i\mu_1 u(x,y,t) - i\mu_2 v(x,y,t)\Big). \tag{5}$$

enables us to express the generating function as $Z(\lambda,\mu_1,\mu_2,x,y,t) = \langle\Theta\rangle$. Using the KPZ equation and its differentiations with respect to x and y and neglecting the viscosity term, in the limit of $\nu \to 0$ before the creation of singularities, we get the following expression for the time evolution of $Z(\lambda,\mu_1,\mu_2,x,y,t)$

$$\begin{aligned} Z_t = {} & i\gamma(t)\lambda Z - i\lambda\frac{\alpha}{2}\langle u^2\Theta\rangle - i\lambda\frac{\alpha}{2}\langle v^2\Theta\rangle \\ & - i\alpha\mu_1\langle uu_x\Theta\rangle - i\alpha\mu_1\langle vv_x\Theta\rangle - i\alpha\mu_2\langle uu_y\Theta\rangle - i\alpha\mu_2\langle vv_y\Theta\rangle \\ & - i\lambda\langle f(x,y,t)\Theta\rangle - i\mu_1\langle f_x(x,y,t)\Theta\rangle - i\mu_2\langle f_y(x,y,t)\Theta\rangle \end{aligned} \tag{6}$$

where $\gamma(t) = \bar{h}_t = \frac{\alpha}{2} < u^2 + v^2 >, k(\mathbf{x}-\mathbf{x}') = 2D_0 D(\mathbf{x}-\mathbf{x}'), k''(0,0) = k_{xx}(0,0) = k_{yy}(0,0)$. To simplify the above expression, by taking the derivative of Z with respect to x and y, and using the homogeneity conditions, we obtain

$$Z_x = \langle(-i\lambda u - i\mu_1 u_x - i\mu_2 u_y)\Theta\rangle = 0 \tag{7}$$

$$Z_y = \langle(-i\lambda v - i\mu_1 v_x - i\mu_2 v_y)\Theta\rangle = 0, \tag{8}$$

from which it can be easily derived

$$i\frac{\partial}{\partial\mu_1}\langle(-i\mu_1 u_x - i\mu_2 u_y)\Theta\rangle = \langle u_x\Theta\rangle - i\mu_1\langle uu_x\Theta\rangle - i\mu_2\langle uu_y\Theta\rangle. \tag{9}$$

From the eqs. (7) and (9) we have

$$\langle u_x\Theta\rangle - i\mu_1\langle uu_x\Theta\rangle - i\mu_2\langle uu_y\Theta\rangle = -\lambda\frac{\partial}{\partial\mu_1}\langle u\Theta\rangle = -i\lambda Z_{\mu_1\mu_1} \tag{10}$$

from which we find the following expression for $-i\alpha\mu_1\langle uu_x\Theta\rangle - i\alpha\mu_2\langle uu_y\Theta\rangle$

$$-i\alpha\mu_1\langle uu_x\Theta\rangle - i\alpha\mu_2\langle uu_y\Theta\rangle = -i\alpha\lambda Z_{\mu_1\mu_1} - \alpha\langle u_x\Theta\rangle. \tag{11}$$

Similarly for $-i\alpha\mu_1\langle vv_x\Theta\rangle - i\alpha\mu_2\langle vv_y\Theta\rangle$ we find

$$-i\alpha\mu_1\langle vv_x\Theta\rangle - i\alpha\mu_2\langle vv_y\Theta\rangle = -i\alpha\lambda Z_{\mu_2\mu_2} - \alpha\langle v_y\Theta\rangle \tag{12}$$

Now from eqs. (11) and (12) and by using the Novikov's theorem the eq.(6) gives

$$\begin{aligned} Z_t &= i\gamma(t)\lambda Z + i\lambda\frac{\alpha}{2}Z_{\mu_1\mu_1} + i\lambda\frac{\alpha}{2}Z_{\mu_2\mu_2} - i\alpha\lambda Z_{\mu_1\mu_1} - i\alpha\lambda Z_{\mu_2\mu_2} \\ &\quad - \alpha\langle u_x\Theta\rangle - \alpha\langle v_y\Theta\rangle - \lambda^2 k(0,0)Z + \mu_1^2 k''(0,0)Z + \mu_2 k''(0,0)Z. \end{aligned} \tag{13}$$

The term $\langle u_x\Theta\rangle$ can be evaluated as follows

$$\begin{aligned} \langle u_x\Theta\rangle &= \frac{i}{\mu_1}\langle\Theta\rangle_x + \frac{i}{\mu_1}\langle(i\lambda u + i\mu_2 v_x)\Theta\rangle \\ &= -i\frac{\lambda}{\mu_1}Z_{\mu_1} - \frac{\mu_2}{\mu_1}\langle v_x\Theta\rangle, \end{aligned} \tag{14}$$

also for $\langle v_y\Theta\rangle$ we have

$$\langle v_y\Theta\rangle = -i\frac{\lambda}{\mu_2}Z_{\mu_2} - \frac{\mu_1}{\mu_2}\langle u_y\Theta\rangle. \tag{15}$$

In the appendix it is proved that $\langle u_y\Theta\rangle = \langle v_x\Theta\rangle = 0$, so the equation governing the time evolution of Z will become

$$\begin{aligned} Z_t &= i\gamma(t)\lambda Z - i\lambda\frac{\alpha}{2}Z_{\mu_1\mu_1} - i\lambda\frac{\alpha}{2}Z_{\mu_2\mu_2} \\ &\quad + i\alpha\frac{\lambda}{\mu_1}Z_{\mu_1} - i\alpha\frac{\lambda}{\mu_2}Z_{\mu_2} \\ &\quad - \lambda^2 k(0,0)Z + \mu_1^2 k''(0,0)Z + \mu_2 k''(0,0)Z. \end{aligned} \tag{16}$$

In what follows we solve this partial differential equation by using flat initial condition, $h(x,y,0) = u(x,y,0) = v(x,y,0) = 0$,which means

$$P(\tilde{h}, u, v, 0) = \delta(\tilde{h})\delta(u)\delta(v), \tag{17}$$

so that the initial condition for the generating function is $Z(\lambda, \mu_1, \mu_2, 0) = 1$.
The solution of the eq.(16) can be factorized as

$$Z(\lambda, \mu_1, \mu_2, t) = F_1(\lambda, \mu_1, t)F_2(\lambda, \mu_2, t)\exp\left(-\lambda^2 k(0,0)t\right), \tag{18}$$

which by inserting eq.(18) in eq.(16) the following equation is obtained

$$\begin{aligned} F_{1t}F_2 + F_1F_{2t} = {} & i\gamma(t)\lambda F_1F_2 - i\lambda\frac{\alpha}{2}F_2F_{1\mu_1\mu_1} - i\lambda\frac{\alpha}{2}F_1F_{2\mu_2\mu_2} \\ & + i\alpha\frac{\lambda}{\mu_1}F_2F_{1\mu_1} - i\alpha\frac{\lambda}{\mu_2}F_1F_{2\mu_2} \\ & - \lambda^2 k(0,0)F_1F_2 + \mu_1^2 k''(0,0)F_1F_2 + \mu_2 k''(0,0)F_1F_2. \end{aligned} \tag{19}$$

On the other hand, $\gamma(t) = \bar{h}_t = \frac{\alpha}{2}\langle u^2 + v^2\rangle$ and $\frac{\alpha}{2}\langle u^2\rangle = \frac{\alpha}{2}\langle v^2\rangle = -\alpha k''(0,0)t$, so eq.(19) can be separated in terms of μ_1 and μ_2 and it can be seen easily that $F_1(\lambda, \mu_1, t)$ and $F_2(\lambda, \mu_2, t)$ are satisfied by the following 1+1-dimensional equation

$$F_t(\lambda, \mu, t) = -i\lambda\frac{\alpha}{2}F_{\mu\mu} + i\alpha\frac{\lambda}{\mu}F_\mu + [\mu^2 k''(0,0) - i\alpha\lambda k''(0,0)t]F, \tag{20}$$

which can be solved exactly with the initial condition $F(\lambda, \mu, 0) = 1$ [14]. The solution of eq.(20) is

$$\begin{aligned} F(\mu, \lambda, t) = {} & (1 - \tanh^2(\sqrt{2ik_{xx}(0,0)\alpha\lambda}t)) \\ & \exp[-\frac{5}{8}\ln(1 - \tanh^4(\sqrt{2ik_{xx}(0,0)\alpha\lambda}t)) \\ & + \frac{5}{4}\tanh^{-1}(\tanh^2(\sqrt{2ik_{xx}(0,0)\alpha\lambda}t)) - \lambda^2 k(0,0)t \\ & - \frac{1}{16}\ln^2(\frac{1 - \tanh(\sqrt{2ik_{xx}(0,0)\alpha\lambda}t)}{1 + \tanh(\sqrt{2ik_{xx}(0,0)\alpha\lambda}t)}) \\ & - \frac{1}{2}i\mu^2\sqrt{\frac{2ik_{xx}(0,0)}{\alpha\lambda}}\tanh(\sqrt{2ik_{xx}(0,0)\alpha\lambda}t)]. \end{aligned} \tag{21}$$

So, the solution of eq.(19) is

$$\begin{aligned} & Z(\lambda, \mu_1, \mu_2, t) = F(\lambda, \mu_1, t)F(\lambda, \mu_2, t)\exp\left(-\lambda^2 k(0,0)t\right) \\ & = (1 - \tanh^2(\sqrt{2ik_{xx}(0,0)\alpha\lambda}t))^2 \exp[-\frac{5}{4}\ln(1 - \tanh^4(\sqrt{2ik_{xx}(0,0)\alpha\lambda}t)) \end{aligned}$$

$$+\frac{5}{2}\tanh^{-1}(\tanh^2(\sqrt{2ik_{xx}(0,0)\alpha\lambda t})) - \lambda^2 k(0,0)t$$
$$-\frac{1}{8}\ln^2(\frac{1-\tanh(\sqrt{2ik_{xx}(0,0)\alpha\lambda t})}{1+\tanh(\sqrt{2ik_{xx}(0,0)\alpha\lambda t})})$$
$$-\frac{1}{2}i(\mu_1^2+\mu_2^2)\sqrt{\frac{2ik_{xx}(0,0)}{\alpha\lambda}}\tanh(\sqrt{2ik_{xx}(0,0)\alpha\lambda t})]. \quad (22)$$

Using the generating function, the probability distribution function (PDF) of height fluctuation can be derived by inverse Fourier transformation as

$$P(\tilde{h},u,v,t) = \int \frac{d\lambda}{2\pi}\frac{d\mu_1}{2\pi}\frac{d\mu_2}{2\pi}\exp\left(i\lambda\tilde{h}+i\mu_1 u+i\mu_2 v\right)Z(\lambda,\mu_1,\mu_2,t). \quad (23)$$

By expanding the solution of the generating function in powers of λ, all the $\langle (h-\bar{h})^n\rangle$ moments can be derived. For instance, the first five moments before the sharp valley formation are

$$\langle\tilde{h}^2\rangle = \left(\frac{k^2(0,0)}{\alpha k''(0,0)}\right)^{2/3}\left[-\frac{2}{3}(\frac{t}{t^*})^4+2\frac{t}{t^*}\right]$$
$$\langle\tilde{h}^3\rangle = -\frac{48}{45}\left(\frac{k^2(0,0)}{\alpha k''(0,0)}\right)(\frac{t}{t^*})^6$$
$$\langle\tilde{h}^4\rangle = \left(\frac{k^2(0,0)}{\alpha k''(0,0)}\right)^{4/3}\left[-\frac{44}{35}(\frac{t}{t^*})^8-8(\frac{t}{t^*})^5+12(\frac{t}{t^*})^2\right]$$
$$\langle\tilde{h}^5\rangle = -\left(\frac{k^2(0,0)}{\alpha k''(0,0)}\right)^{5/3}\left[\frac{1216}{945}(\frac{t}{t^*})^{10}+\frac{64}{3}(\frac{t}{t^*})^7\right] \quad (24)$$

where $t_* = (\frac{k(0,0)}{\alpha^2 k''^2(0,0)})^{1/3}$. An important information content of the derived exact form is that they determine the time scale of the singularity formations. One should first check the realisability condition, i.e. $P(h-\bar{h},t)\geq 0$. In fact the moment relations above indicate that different even order moments will get *negative* in some distinct characteristic time scales. A closer inspection into the even moment relations reveals that the higher the moments are, the smaller their characteristic time scales become. So the rate of decreasing tends to $t_c=\frac{1}{4}t_*$ for very large even moments asymptotically, where $t_* = (\frac{k(0,0)}{\alpha^2 k''^2(0,0)})^{1/3} = (\frac{\pi}{8\alpha^2 D_0})^{1/3}\sigma^2$. Therefore, it is concluded that at this time the right tail of the probability distribution function (PDF) starts to become negative, which is the reminiscent of singularity creation. Physically, the connection between the negativity of the right tail and the singularity formation time scale is related to a multi-valued solution of KPZ (Burgers) equation for $t\geq t_c$. It should

be notified that eq.(16) preserves the normalisability, i.e. $Z(0,0,0,t) = 1$, equivalent to $\int_{-\infty}^{+\infty} P(h-\bar{h},u,v;t)d(h-\bar{h})dudv = 1$ for every time t. So the PDF of $h-\bar{h}$ and its derivatives are always normalizable to unity. In the $\nu = 0$ limit, for time scales greater than t_c , the height field will become multi-valued in the valleys (the shock region), which is related to the $P(h-\bar{h})$ left tail . The multiplicity of the height field will increase the probability measure in the PDF. Therefore, to compensate the exceeded measure related to the multi-valued solutions the right tail of the PDF tails should become negative. The singularities in the limit $\nu \to 0$ can be constructed from multi-valued solutions of the KPZ equation with $\nu = 0$ by Maxwell cutting rule [13], which makes the discontinuity in the derivative of the height filed. When $t > t_c$, the contribution of the relaxation term in the limit of vanishing diffusion coefficient should also be considered in order to find a realisable probability density function of height field. In other words, the disregardance of the PDF equation's diffusion term is only valid up to the time scales in which the singularities have not been developed yet.

Taking into account $\alpha > 0$ and $k''(0,0) < 0$, the odd order moments will be positive in time scales $t < t_c$ and the probability density function $P(h-\bar{h},t)$ is positively skewed. Therefore, the probability distribution functions of height field will have a non zero skewness through its time evolution at least before the singularity formations time scale.The inverse Fourier transform of the generating function has been performed numerically to obtain the PDF $P(h-\bar{h})$ form. To demonstrate the time scale of the singularity formation the PDF time evolution has been sketched numerically, Fig(1). As the system evolves in time, the formation of the first singularities leads to the right tail negativity in the PDF.

To summarise, we analyse the fully nonlinear KPZ equation in 2+1 dimensions forced with a Gaussian noise which is white in time and short range correlated in space. In the non-stationary regime when the sharp valley structures are not yet developed we find an exact form for the generating function of the joint fluctuations of height and height gradient. We determine the time scale of the sharp valley formation and the exact functional form of the time dependence in the height difference moments at any given order. In this paper, we have used the initial condition $h(x,0) = 0$ and $u(x,0) = 0$. This method enables us to determine the finite time scale of singularity formation in terms of statistical properties of the initial conditions. This calculation will be dealt with elsewhere. At this level we would remark on the stationary state of the KPZ equation in the strong coupling limit. Using the numerical results recently, it is shown that convergence to the statistical steady state is reached after a few turnover times [16]. Therefore, in the stationary state the singularities will be fully developed and, therefore, at large t or $t \to \infty$ we should take into account the relaxation contribution in the limit $\nu \to 0$. Contribution of the relaxation term in the rhs

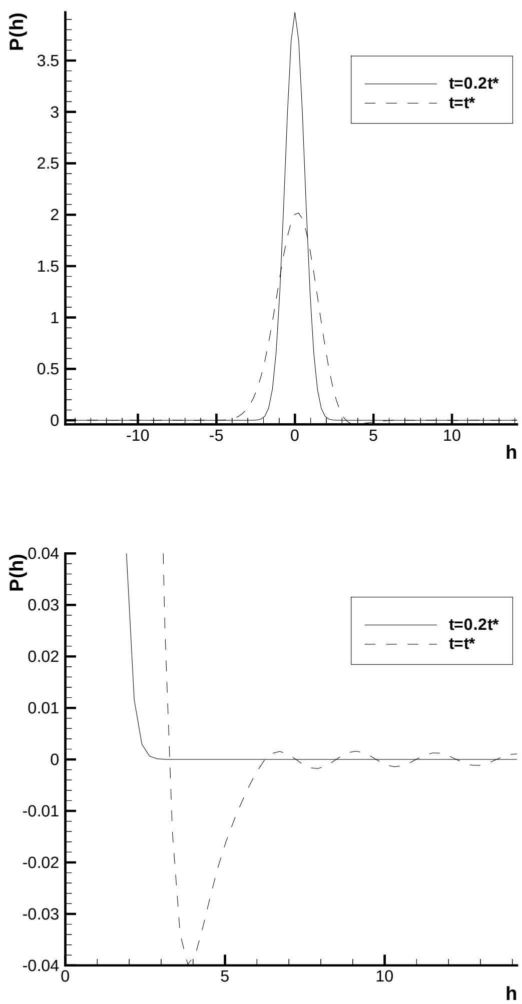

FIG. 1. In the LHS graph the time evolution of PDF of $h - \bar{h}$ before singularity formation at $\frac{2}{10}t_*$ and t_* is numerically obtained. RHS graph shows the right tails of the PDF of $h - \bar{h}$ for $\frac{2}{10}t_*$ and t_* corresponding to before and after singularities formation which are numerically calculated.

of the eq.(6) is,

$$\lim_{\nu\to 0}\{-i\nu\mu_1\langle\ \nabla^2 u\Theta\rangle - i\nu\mu_2\langle\ \nabla^2 v\Theta\rangle\}. \tag{25}$$

In [14] we have calculated the finite contribution of this term in 1+1 dimensions and showed that it plays an important role in statistical theory of KPZ equation in the stationary state. It is also shown that all of the amplitudes of the moments $<(h-\bar{h})^n>$ can be expressed in term of the relaxation term in the limit $\nu \to 0$. In 2+1 dimension calculation of finite contribution of the relaxation term is more complex (because of complex structure of the singularities) and it is left for a future work. We believe that the analysis followed in that paper is also suitable for the zero temperature limit in the problem of directed polymer in the random potential with short range correlations [10]. In the same direction the present method is applicable to the *strong coupling* regime of KPZ equation in higher dimensions ($d > 2$) which is definitely an important step.

Acknowledgement
We thank J. Davoudi, F. Arash, M. Khorrami, A. Reza-Khani and A. Tabei for useful discussions. F.S and A.A.M. were partially supported by the Institute for Studies in Physics and Mathematics (IPM).

III. APPENDIX

In this appendix we present another way of deriving the height moments by introducing the second spatial derivatives of height field. Defining $\Theta(\lambda,\mu_1,\mu_2,\eta_1,\eta_2,\eta_3,x,y,t)$ as

$$\begin{aligned}\Theta(\lambda,\mu_1,\mu_2,\eta_1,\eta_2,\eta_3,x,y,t) = \exp\{&-i\lambda(h(x,y,t)-\bar{h}(t)) - i\mu_1 u(x,y,t) - i\mu_2 v(x,y,t)\\ &-i\eta_1 w(x,y,t) - i\eta_2 s(x,y,t) - i\eta_3 q(x,y,t)\},\end{aligned} \tag{26}$$

the generating function is defined as

$$Z(\lambda,\mu_1,\mu_2,\eta_1,\eta_2,\eta_3,x,y,t) = \langle\Theta\rangle, \tag{27}$$

where $u(x,y,t) = h_x(x,y,t)$,$v(x,y,t) = h_y(x,y,t)$,$w(x,y,t) = h_{xx}(x,y,t)$, $s(x,y,t) = h_{xy}(x,y,t)$ and $q(x,y,t) = h_{yy}(x,y,t)$.

The evolution of $h(x,y,t), u(x,y,t), v(x,y,t), w(x,y,t), s(x,y,t), q(x,y,t)$ is given by the following equations

$$h_t = \frac{\alpha}{2}(u^2+v^2) + f(x,y,t) \tag{28}$$

$$u_t = \alpha(uw+vs) + f_x(x,y,t) \tag{29}$$

$$v_t = \alpha(us + vq) + f_y(x, y, t) \tag{30}$$

$$w_t = \alpha(w^2 + s^2 + uw_x + vs_x) + f_{xx}(x, y, t) \tag{31}$$

$$s_t = \alpha(ws + qs + vs_y uw_y) + f_{xy}(x, y, t) \tag{32}$$

$$q_t = \alpha(s^2 + q^2 + us_y + vq_y) + f_{yy}. \tag{33}$$

It follows from the above equations that the generating function Z is the solution of the following equation

$$\begin{aligned} Z_t = {} & i\gamma(t)\lambda Z - i\lambda\frac{\alpha}{2}Z_{\mu_1\mu_1} - i\lambda\frac{\alpha}{2}Z_{\mu_2\mu_2} + i\alpha Z_{\mu_1 x} + i\alpha Z_{\mu_2 y} \\ & - i\alpha Z_{\eta_1} - i\alpha Z_{\eta_3} + i\alpha\eta_1 Z_{\eta_1\eta_1} + i\alpha\eta_1 Z_{\eta_2\eta_2} \\ & + i\alpha\eta_2 Z_{\eta_1\eta_2} + i\alpha\eta_2 Z_{\eta_2\eta_3} + i\alpha\eta_3 Z_{\eta_2\eta_2} + i\alpha\eta_3 Z_{\eta_3\eta_3} \\ & - i\lambda\langle f(x, y, t)\Theta\rangle - i\mu_1\langle f_x(x, y, t)\Theta\rangle - i\mu_2\langle f_y(x, y, t)\Theta\rangle \\ & - i\eta_1\langle f_{xx}(x, y, t)\Theta\rangle - i\eta_2\langle f_{xy}(x, y, t)\Theta\rangle - i\eta_3\langle f_{yy}(x, y, t)\Theta\rangle, \end{aligned} \tag{34}$$

in which $\gamma(t)$ is defined as $\gamma(t) = \bar{h}_t$ and the following identities have been used

$$\eta_1\langle w_x\Theta\rangle + \eta_2\langle s_x\Theta\rangle + \eta_3\langle q_x\Theta\rangle = iZ_x - i\lambda Z_{\mu_1} - i\mu_1 Z_{\eta_1} - i\mu_2 Z_{\eta_2} \tag{35}$$

$$\eta_1\langle w_y\Theta\rangle + \eta_2\langle s_y\Theta\rangle + \eta_3\langle q_y\Theta\rangle = iZ_y - i\lambda Z_{\mu_2} - i\mu_1 Z_{\eta_2} - i\mu_2 Z_{\eta_3}. \tag{36}$$

Now, using Novikov's theorem we find

$$\langle f(x, y, t)\Theta\rangle = -i\lambda k(0, 0)Z - i\eta_1 k_{xx}(0, 0)Z - i\eta_3 k_{xx}(0, 0)Z \tag{37}$$

$$\langle f_x(x, y, t)\Theta\rangle = i\mu_1 k_{xx}(0, 0)Z \tag{38}$$

$$\langle f_y(x, y, t)\Theta\rangle = i\mu_2 k_{xx}(0, 0)Z \tag{39}$$

$$\langle f_{xx}(x, y, t)\Theta\rangle = -i\lambda k_{xx}(0, 0)Z - i\eta_1 k_{xxxx}(0, 0)Z - i\eta_3 k_{xxxx}(0, 0)Z \tag{40}$$

$$\langle f_{xy}(x, y, t)\Theta\rangle = -i\eta_2 k_{xxxx}(0, 0)Z \tag{41}$$

$$\langle f_{yy}(x, y, t)\Theta\rangle = -i\lambda k_{xx}(0, 0)Z - i\eta_1 k_{xxxx}(0, 0)Z - i\eta_3 k_{xxxx}(0, 0)Z, \tag{42}$$

where $k(x - x', y - y') = 2D_0 D(x - x', y - y'), k(0, 0) = \frac{2D_0}{\pi\sigma^2}, k_{xx}(0, 0) = k_{yy}(0, 0) = -\frac{4D_0}{\pi\sigma^4}$ and $k_x(0, 0) = k_y(0, 0) = 0$. So we have

$$\begin{aligned} Z_t = {} & i\gamma(t)\lambda Z - i\lambda\frac{\alpha}{2}Z_{\mu_1\mu_1} - i\lambda\frac{\alpha}{2}Z_{\mu_2\mu_2} + i\alpha Z_{\mu_1 x} + i\alpha Z_{\mu_2 y} \\ & - i\alpha Z_{\eta_1} - i\alpha Z_{\eta_3} + i\alpha\eta_1 Z_{\eta_1\eta_1} + i\alpha\eta_1 Z_{\eta_2\eta_2} \\ & + i\alpha\eta_2 Z_{\eta_1\eta_2} + i\alpha\eta_2 Z_{\eta_2\eta_3} + i\alpha\eta_3 Z_{\eta_2\eta_2} + i\alpha\eta_3 Z_{\eta_3\eta_3} \\ & - \lambda^2 k(0, 0)Z + (\mu_1^2 + \mu_2^2 - 2\lambda\eta_1 - 2\lambda\eta_3)k_{xx}(0, 0)Z \\ & - (\eta_1^2 + \eta_2^2 + \eta_3^2 + 2\eta_1\eta_3)k_{xxxx}(0, 0)Z. \end{aligned} \tag{43}$$

Assuming statistical homogeneity ($Z_x = 0, Z_y = 0$) and defining $P(\tilde{h}, u, v, w, s, q, t)$ as the joint probability density function of $\tilde{h}, u, v, w, s$ and q, one can construct the PDF as the Fourier transform of the generating function Z with respect to $\lambda, \mu_1, \mu_2, \eta_1, \eta_2, \eta_3$

$$
\begin{aligned}
&P(\tilde{h}, u, v, w, s, q, t) = \int \frac{d\lambda}{2\pi}\frac{d\mu_1}{2\pi}\frac{d\mu_2}{2\pi}\frac{d\eta_1}{2\pi}\frac{d\eta_2}{2\pi}\frac{d\eta_3}{2\pi}\{ \\
&\exp\Big(i\lambda\tilde{h} + i\mu_1 u + i\mu_2 v + i\eta_1 w + i\eta_2 s + i\eta_3 q\Big) Z(\lambda, \mu_1, \mu_2, \eta_1, \eta_2, \eta_3, t)\}.
\end{aligned} \tag{44}
$$

From the eqs. (43) and (44) the equation governing the evolution of $P(\tilde{h}, u, v, w, s, q, t)$ can be derived, which is

$$
\begin{aligned}
P_t &= \gamma(t)P_{\tilde{h}} + \frac{\alpha}{2}(u^2 + v^2)P_{\tilde{h}} - 4\alpha w P - 4\alpha q P \\
&- \alpha w^2 P_w - \alpha q^2 P_q - \alpha s^2 P_w - \alpha s^2 P_q - \alpha w s P_s - \alpha q s P_s \\
&+ k(0,0)P_{\tilde{h}\tilde{h}} + 2k_{xx}(0,0)P_{\tilde{h}w} + 2k_{xx}(0,0)P_{\tilde{h}q} - k_{xx}(0,0)P_{uu} - k_{xx}(0,0)P_{vv} \\
&+ k_{xxxx}(0,0)P_{ww} + k_{xxxx}(0,0)P_{ss} + k_{xxxx}(0,0)P_{qq} + 2k_{xxxx}(0,0)P_{qw}.
\end{aligned} \tag{45}
$$

From the eq.(45), it is easy to see that the moments $\langle \tilde{h}^{n_0} u^{n_1} v^{n_2} w^{n_3} s^{n_4} q^{n_5} \rangle$ satisfy the following equation

$$
\begin{aligned}
&\frac{d}{dt}\langle\tilde{h}^{n_0}u^{n_1}v^{n_2}w^{n_3}s^{n_4}q^{n_5}\rangle = -n_0\gamma(t)\langle\tilde{h}^{n_0-1}u^{n_1}v^{n_2}w^{n_3}s^{n_4}q^{n_5}\rangle - \frac{\alpha n_0}{2}\langle\tilde{h}^{n_0-1}u^{n_1+2}v^{n_2}w^{n_3}s^{n_4}q^{n_5}\rangle \\
&-\frac{\alpha n_0}{2}\langle\tilde{h}^{n_0-1}u^{n_1}v^{n_2+2}w^{n_3}s^{n_4}q^{n_5}\rangle - 4\alpha\langle\tilde{h}^{n_0}u^{n_1}v^{n_2}w^{n_3+1}s^{n_4}q^{n_5}\rangle \\
&-4\alpha\langle\tilde{h}^{n_0}u^{n_1}v^{n_2}w^{n_3}s^{n_4}q^{n_5+1}\rangle + \alpha(n_3+2)\langle\tilde{h}^{n_0}u^{n_1}v^{n_2}w^{n_3+1}s^{n_4}q^{n_5}\rangle \\
&+\alpha n_3\langle\tilde{h}^{n_0}u^{n_1}v^{n_2}w^{n_3-1}s^{n_4+2}q^{n_5}\rangle + \alpha(n_4+1)\langle\tilde{h}^{n_0}u^{n_1}v^{n_2}w^{n_3+1}s^{n_4}q^{n_5}\rangle \\
&+\alpha(n_4+1)\langle\tilde{h}^{n_0}u^{n_1}v^{n_2}w^{n_3}s^{n_4}q^{n_5+1}\rangle + \alpha(n_5+2)\langle\tilde{h}^{n_0}u^{n_1}v^{n_2}w^{n_3}s^{n_4}q^{n_5+1}\rangle \\
&+n_5(n_5-1)k_{xxxx}(0,0)\langle\tilde{h}^{n_0}u^{n_1}v^{n_2}w^{n_3}s^{n_4}q^{n_5-2}\rangle + 2n_3n_5k_{xxxx}(0,0)\langle\tilde{h}^{n_0}u^{n_1}v^{n_2}w^{n_3-1}s^{n_4}q^{n_5-1}\rangle \\
&+2n_0n_3k_{xx}(0,0)\langle\tilde{h}^{n_0-1}u^{n_1}v^{n_2}w^{n_3-1}s^{n_4}q^{n_5}\rangle + 2n_0n_5k_{xx}(0,0)\langle\tilde{h}^{n_0-1}u^{n_1}v^{n_2}w^{n_3}s^{n_4}q^{n_5-1}\rangle \\
&-n_1(n_1-1)k_{xx}(0,0)\langle\tilde{h}^{n_0}u^{n_1-2}v^{n_2}w^{n_3}s^{n_4}q^{n_5}\rangle - n_2(n_2-1)k_{xx}(0,0)\langle\tilde{h}^{n_0}u^{n_1}v^{n_2-2}w^{n_3}s^{n_4}q^{n_5}\rangle \\
&+n_3(n_3-1)k_{xxxx}(0,0)\langle\tilde{h}^{n_0}u^{n_1}v^{n_2}w^{n_3-2}s^{n_4}q^{n_5}\rangle + n_4(n_4-1)k_{xxxx}(0,0)\langle\tilde{h}^{n_0}u^{n_1}v^{n_2}w^{n_3}s^{n_4-2}q^{n_5}\rangle \\
&+\alpha n_5\langle\tilde{h}^{n_0}u^{n_1}v^{n_2}w^{n_3}s^{n_4+2}q^{n_5-1}\rangle + n_0(n_0-1)k(0,0)\langle\tilde{h}^{n_0-2}u^{n_1}v^{n_2}w^{n_3}s^{n_4}q^{n_5}\rangle. \qquad (46)
\end{aligned}
$$

By substituting different sort of values for $n_0, n_1, n_2, n_3, n_4, n_5$ we can find some coupled differential equations for different moments. For example, if we take $n_0 = n_1 = \cdots = n_5 = 0$ we have

$$
< w > + < q >= 0 \qquad (47)
$$

or

$$
< \nabla \cdot \mathbf{u} >= \mathbf{0} \qquad (48)
$$

where this is the same as the statistical hemogeneity condition. For $n_0 = 1, n_1 = n_2 = n_3 = n_4 = n_5 = 0$ we find

$$
< \tilde{h}w > + < \tilde{h}q >= - < u^2 > - < v^2 > \qquad (49)
$$

and for $n_0 = 0, n_1 = 1, n_2 = n_3 = n_4 = n_5 = 0$ we find

$$
< uw > + < uq >= 0, \qquad (50)
$$

and if we assume the statistical hemogeneity we have

$$
< uw >=< uu_x >= \frac{1}{2} < u^2 >_x = 0. \qquad (51)
$$

So $< uw >= 0$ and then $< uq >= 0$, also for $n_0 = n_1 = 0, n_2 = 1, n_3 = n_4 = n_5 = 0$ we find $< vq >= 0, < vw >= 0$.

For $n_0 = 0, n_1 = 2, n_2 = n_3 = n_4 = n_5 = 0$ we find

$$
\frac{d}{dt} < u^2 >= -\alpha < u^2 w > -\alpha < u^2 q > -2k_{xx}(0,0) \qquad (52)
$$

and also we have $< u^2 w >= \frac{1}{3} < u^3 >_x = 0$ by statistical hemogeneity and also $< u^2 q >=< u^2 v >_y -2 < uvs >$, $< u^2 v >_y = 0$, so $< u^2 q >= -2 < uvs >$ so that we have

$$\frac{d}{dt} < u^2 >= 2\alpha < uvs > -2k_{xx}(0,0). \tag{53}$$

The corresponding differential equation for $< uvs >$ is

$$\frac{d}{dt} < uvs >= 0. \tag{54}$$

If we assume that at $t = 0$ the surface is flat so all the moments at $t = 0$ are zero,so that $< uvs >= 0$ and then we find

$$< u^2 >= -2k_{xx}(0,0)t. \tag{55}$$

Similar calculations give

$$< v^2 >= -2k_{xx}(0,0)t. \tag{56}$$

By focusing on the differential equation of the PDF it is deduced that this equation is invariant under $u \to -u$ and $v \to -v$ which is a consequence of the inversion and reflection symmetry of the KPZ equation. So it will result in the vanishing of the u and v odd moments

$$\langle u^{2k+1} \rangle = \langle v^{2k+1} \rangle = 0. \tag{57}$$

Using eq.(46) the even moments of u and v can be calculated. For example, for $\langle u^4 \rangle$ we have

$$\frac{d}{dt}\langle u^4 \rangle = -\alpha \langle u^4 w \rangle - \alpha \langle u^4 q \rangle - 12k_{xx}(0)\langle u^2 \rangle. \tag{58}$$

On the other hand $\langle u^4 w \rangle$ can be written as

$$\langle u^4 u_x \rangle = \frac{1}{5}\langle u^5 \rangle_x, \tag{59}$$

which is zero by homogeneity, also for $\langle u^4 q \rangle$ we have

$$\langle u^4 v_y \rangle = \langle u^4 v \rangle_y - 4\langle u^3 vs \rangle = -4\langle u^3 vs \rangle \tag{60}$$

in which $\langle u^4 v \rangle_y$ is zero by homogeneity. The differential equation for $\langle u^3 vs \rangle$ is

$$\frac{d}{dt}\langle u^3 vs \rangle = 0, \tag{61}$$

so $\langle u^3 vs \rangle = 0$. Therefore, it is obtained

$$\langle u^4 \rangle = 12k_{xx}^2(0,0)t^2, \tag{62}$$

and we have the same for $\langle v^4\rangle$. Also it can be found that $\langle u^2v^2\rangle = 4k_{xx}^2(0,0)t^2$. By continuing the above method all the moments $\langle u^n v^m\rangle$ can be deduced.

On the other hand, the above calculations show that all the mixed moments $\langle u^n v^m s\rangle$ are zero. In the following some of the moments of u and v and their combinations have been given

$$\langle u^6\rangle = \langle v^6\rangle = -120k_{xx}^3(0,0)t^3 \tag{63}$$
$$\langle u^2v^4\rangle = \langle u^4v^2\rangle = -24k_{xx}^3(0,0)t^3 \tag{64}$$
$$\langle u^8\rangle = \langle v^8\rangle = 1680k_{xx}^4(0,0)t^4 \tag{65}$$
$$\langle u^6v^2\rangle = \langle u^2v^6\rangle = 240k_{xx}^4(0,0)t^4 \tag{66}$$
$$\langle u^4v^4\rangle = 144k_{xx}^4(0,0)t^4, etc. \tag{67}$$

We aim to calculate all the momentes of $\tilde{h} = h - \bar{h}$. Putting $n_0 = 2, n_1 = n_2 = \cdots = 0$ in the eq.(46), we find

$$\frac{d}{dt}\langle\tilde{h}^2\rangle = -\alpha\langle\tilde{h}u^2\rangle - \alpha\langle\tilde{h}v^2\rangle - \alpha\langle\tilde{h}^2 w\rangle - \alpha\langle\tilde{h}^2 q\rangle + 2k(0,0). \tag{68}$$

We know that $\langle\tilde{h}^2 w\rangle = \langle\tilde{h}^2 u\rangle_x - 2\langle\tilde{h}u^2\rangle$ and because of the homogeneity $\langle\tilde{h}^2 u\rangle_x = 0$, so we have

$$\langle\tilde{h}^2 w\rangle = -2\langle\tilde{h}u^2\rangle, \tag{69}$$

and also

$$\langle\tilde{h}^2 q\rangle = -2\langle\tilde{h}v^2\rangle, \tag{70}$$

then

$$\frac{d}{dt}\langle\tilde{h}^2\rangle = \alpha(\langle\tilde{h}u^2\rangle + \langle\tilde{h}v^2\rangle) + 2k(0,0). \tag{71}$$

As it can be seen from eq.(71) we need the moments $\langle\tilde{h}u^2\rangle$ and $\langle\tilde{h}v^2\rangle$ to find $\langle\tilde{h}^2\rangle$, which the related differential equations are

$$\frac{d}{dt}\langle\tilde{h}u^2\rangle = -\gamma(t)\langle u^2\rangle - \frac{\alpha}{2}\langle u^4\rangle - \frac{\alpha}{2}\langle u^2v^2\rangle - \alpha\langle\tilde{h}u^2 w\rangle - \alpha\langle\tilde{h}u^2 q\rangle \tag{72}$$
$$\frac{d}{dt}\langle\tilde{h}v^2\rangle = -\gamma(t)\langle v^2\rangle - \frac{\alpha}{2}\langle v^4\rangle - \frac{\alpha}{2}\langle u^2v^2\rangle - \alpha\langle\tilde{h}v^2 w\rangle - \alpha\langle\tilde{h}v^2 q\rangle. \tag{73}$$

By using of the statistical homogeneity the last two terms of the above equations can be converted as

$$\begin{aligned}
\langle \tilde{h}u^2w\rangle &= -\frac{1}{3}\langle u^4\rangle \\
\langle \tilde{h}v^2q\rangle &= -\frac{1}{3}\langle v^4\rangle \\
\langle \tilde{h}u^2q\rangle &= -\langle u^2v^2\rangle - 2\langle \tilde{h}uvs\rangle \\
\langle \tilde{h}v^2w\rangle &= -\langle u^2v^2\rangle - 2\langle \tilde{h}uvs\rangle. \qquad (74)
\end{aligned}$$

The above relations result in

$$\begin{aligned}
\frac{d}{dt}(\langle \tilde{h}u^2\rangle + \langle \tilde{h}v^2\rangle &= -\gamma(t)(\langle u^2+v^2\rangle) - \frac{\alpha}{6}(\langle u^4+v^4\rangle) \\
&+ \alpha\langle u^2v^2\rangle + 4\alpha\langle \tilde{h}uvs\rangle. \qquad (75)
\end{aligned}$$

Obtaining the differential equation for $\langle \tilde{h}uvs\rangle$ we get

$$\frac{d}{dt}\langle \tilde{h}uvs\rangle = 0, \qquad (76)$$

which results in $\langle \tilde{h}uvs\rangle = 0$. Also it is easy to see that all the moments $\langle \tilde{h}^n u^m v^p s\rangle$ are zero. This fantastic result helps us to find all the $\langle \tilde{h}^n\rangle$ moments. Now by substituting $\gamma(t) = \bar{h}_t = \frac{\alpha}{2}(\langle u^2+v^2\rangle) = -2\alpha k_{xx}(0,0)t$, $\langle u^4\rangle$, $\langle v^4\rangle$ and $\langle u^2v^2\rangle$ in eq.(75) we obtain

$$\langle \tilde{h}u^2\rangle + \langle \tilde{h}v^2\rangle = -\frac{8}{3}\alpha k_{xx}^2(0,0)t^3, \qquad (77)$$

which finally gives

$$\langle \tilde{h}^2\rangle = -\frac{2}{3}\alpha^2 k_{xx}^2(0,0)t^4 + 2k(0,0)t. \qquad (78)$$

Now we begin to calculate the moment $\langle \tilde{h}^3\rangle$. Inserting $n_0 = 3, n_1 = n_2 = \cdots = n_5 = 0$ in eq.(46) we get

$$\frac{d}{dt}\langle \tilde{h}^3\rangle = -3\gamma(t)\langle \tilde{h}^2\rangle - \frac{3}{2}\alpha(\langle \tilde{h}^2u^2\rangle + \langle \tilde{h}^2v^2\rangle) - \alpha\langle \tilde{h}^3w\rangle - \alpha\langle \tilde{h}^3q\rangle, \qquad (79)$$

and again by using statisiacal homogeneity it can be shown that

$$\langle \tilde{h}^3w\rangle = -3\langle \tilde{h}^2u^2\rangle \qquad (80)$$

$$\langle \tilde{h}^3q\rangle = -3\langle \tilde{h}^2v^2\rangle. \qquad (81)$$

So we have

$$\frac{d}{dt}\langle \tilde{h}^3\rangle = -3\gamma(t)\langle \tilde{h}^2\rangle + \frac{3}{2}\alpha(\langle \tilde{h}^2u^2\rangle + \langle \tilde{h}^2v^2\rangle). \qquad (82)$$

To calculate $\langle \tilde{h}^3\rangle$ we need $\langle \tilde{h}^2\rangle$, $(\langle \tilde{h}^2u^2 + \tilde{h}^2v^2\rangle)$. $\langle \tilde{h}^2\rangle$ has been calculated above, so we will obtain $\langle \tilde{h}^2u^2 + \tilde{h}^2v^2\rangle$ using the corresponding differential equation as follows,

$$\frac{d}{dt}(\langle\tilde{h}^2u^2\rangle+\langle\tilde{h}^2v^2\rangle=-2\gamma(t)(\langle\tilde{h}u^2+\tilde{h}v^2\rangle)-\alpha(\langle\tilde{h}u^4+\tilde{h}v^4\rangle)-2\alpha\langle\tilde{h}u^2v^2\rangle-\alpha(\langle\tilde{h}^2u^2w+\tilde{h}^2v^2w\rangle)$$
$$-\alpha(\langle\tilde{h}^2u^2q+\tilde{h}^2v^2q\rangle)+2k(0,0)(\langle u^2+v^2\rangle)-4k_{xx}(0,0)\langle\tilde{h}^2\rangle. \tag{83}$$

As before, we easily get

$$\begin{aligned}
\langle\tilde{h}^2u^2w\rangle &= -\frac{2}{3}\langle\tilde{h}u^4\rangle\\
\langle\tilde{h}^2v^2q\rangle &= -\frac{2}{3}\langle\tilde{h}v^4\rangle\\
\langle\tilde{h}^2u^2q\rangle &= -2\langle\tilde{h}u^2v^2\rangle-2\langle\tilde{h}^2uvs\rangle\\
\langle\tilde{h}^2v^2w\rangle &= -2\langle\tilde{h}u^2v^2\rangle-2\langle\tilde{h}^2uvs\rangle.
\end{aligned} \tag{84}$$

As discussed before, it is easy to show that the moment $\langle\tilde{h}^2uvs\rangle$ is zero. To prove this we write the corresponding differential equation

$$\frac{d}{dt}\langle\tilde{h}^2uvs\rangle=-\alpha(\langle\tilde{h}u^3vs\rangle+\langle\tilde{h}uv^3s\rangle), \tag{85}$$

and again by trying to write the differential eqautions for $\langle\tilde{h}u^3vs\rangle$ and $\langle\tilde{h}uv^3s\rangle$ we obtain

$$\begin{aligned}
\frac{d}{dt}\langle\tilde{h}u^3vs\rangle &= 0\\
\frac{d}{dt}\langle\tilde{h}uv^3s\rangle &= 0
\end{aligned} \tag{86}$$

which results in $\langle\tilde{h}u^3vs\rangle=\langle\tilde{h}uv^3s\rangle=0$, therefore $\langle\tilde{h}^2uvs\rangle=0$. Now $\langle\tilde{h}u^4\rangle$, $\langle\tilde{h}v^4\rangle$ and $\langle\tilde{h}u^2v^2\rangle$ should be found. The relating differential equation for $\langle\tilde{h}u^4+\tilde{h}v^4\rangle$ is

$$\frac{d}{dt}(\langle\tilde{h}u^4\rangle+\langle\tilde{h}v^4\rangle=-\gamma(t)(\langle u^4+v^4\rangle)-\frac{\alpha}{2}(\langle u^6+v^6\rangle)-\frac{\alpha}{2}(\langle u^4v^2+u^2v^4\rangle)$$
$$-\alpha(\langle\tilde{h}u^4w+\tilde{h}v^4w\rangle)-\alpha(\langle\tilde{h}u^4q+\tilde{h}v^4q\rangle)-12k_{xx}(0,0)(\langle\tilde{h}u^2+\tilde{h}v^2\rangle). \tag{87}$$

As before the following identities are held

$$\begin{aligned}
\langle\tilde{h}u^4w\rangle &= -\frac{1}{5}\langle u^6\rangle\\
\langle\tilde{h}u^4q\rangle &= -\langle u^4v^2\rangle\\
\langle\tilde{h}v^4q\rangle &= -\frac{1}{5}\langle v^6\rangle\\
\langle\tilde{h}v^4w\rangle &= -\langle u^2v^4\rangle,
\end{aligned} \tag{88}$$

so we have

$$\frac{d}{dt}(\langle\tilde{h}u^4\rangle+\langle\tilde{h}v^4\rangle=-\gamma(t)(\langle u^4+v^4\rangle)-\frac{3\alpha}{10}(\langle u^6+v^6\rangle)+\frac{\alpha}{2}(\langle u^4v^2+u^2v^4\rangle)$$
$$-12k_{xx}(0,0)(\langle\tilde{h}u^2+\tilde{h}v^2\rangle). \tag{89}$$

Substituting the expressions for $\gamma(t)$,$\langle u^4\rangle$, $\langle v^4\rangle$, $\langle u^6\rangle$, $\langle v^6\rangle$, $\langle u^4v^2\rangle$, $\langle u^2v^4\rangle$ and $(\langle\tilde{h}u^2+\tilde{h}v^2\rangle)$ we find

$$\langle\tilde{h}u^4+\tilde{h}v^4\rangle = 32\alpha k_{xx}(0,0)^3t^4. \tag{90}$$

The corresponding differential equation for $\langle\tilde{h}u^2v^2\rangle$ is

$$\begin{aligned}\frac{d}{dt}(\langle\tilde{h}u^2v^2\rangle &= -\gamma(t)(\langle u^2v^2\rangle) - \frac{\alpha}{2}(\langle u^4v^2+u^2v^4\rangle) \\ &- \alpha(\langle\tilde{h}u^2v^2w+\tilde{h}u^2v^2q\rangle) - 2k_{xx}(0,0)(\langle\tilde{h}u^2+\tilde{h}v^2\rangle).\end{aligned} \tag{91}$$

As before the following identities are held

$$\begin{aligned}\langle\tilde{h}u^2v^2w\rangle &= -\frac{1}{3}\langle u^4v^2\rangle \\ \langle\tilde{h}u^2v^2q\rangle &= -\frac{1}{3}\langle u^2v^4\rangle,\end{aligned} \tag{92}$$

then by inserting the known moments we find

$$\langle\tilde{h}u^2v^2\rangle = \frac{16}{3}\alpha k_{xx}^3(0,0)t^4. \tag{93}$$

The above results get

$$\langle\tilde{h}^2u^2\rangle + \langle\tilde{h}^2v^2\rangle = -\frac{8}{3}\alpha^2k_{xx}^3(0,0)t^5 - 8k(0,0)k_{xx}(0,0)t^2, \tag{94}$$

which finally results in

$$\langle\tilde{h}^3\rangle = -\frac{48}{45}\alpha^3k_{xx}^3(0,0)t^6. \tag{95}$$

By continuing the above proccedure all the $\langle\tilde{h}^n\rangle$ moments can be derived. Some of these moments are listed bellow

$$\langle\tilde{h}^4\rangle = -\frac{44}{35}\alpha^4k_{xx}^4(0,0)t^8 - 8\alpha^2k(0,0)k_{xx}^2(0,0)t^5 + 12k^2(0,0)t^2 \tag{96}$$

$$\langle\tilde{h}^5\rangle = -\frac{1216}{945}\alpha^5k_{xx}^5(0,0)t^{10} - \frac{64}{3}\alpha^3k(0,0)k_{xx}^3(0,0)t^7, \tag{97}$$

which are the same as the results that we derived by expanding the generating function. Results of this appendix show that any moment containing the first power of s vanishes. Indeed, one can prove the following identity

$$\langle se^{-i(\lambda\tilde{h}+\mu_1u+\mu_2v)}\rangle = 0. \tag{98}$$

By expanding the exponential in the above expression one finds

$$\langle s e^{-i(\lambda\tilde{h}+\mu_1 u+\mu_2 v)}\rangle = \sum_{n,m,p} \frac{i^{(n+m+p)}\lambda^n \mu_1^m \mu_2^p}{n!m!p!}\langle s\tilde{h}^n u^m v^p\rangle. \tag{99}$$

Now setting $n_3 = n_5 = 0$ and $n_4 = 1$ in eq.(46) we get the following equation for $\langle s\tilde{h}^n u^m v^p\rangle$

$$\begin{aligned}\frac{d}{dt}\langle\tilde{h}^n u^m v^p s\rangle = & -n\gamma(t)\langle\tilde{h}^{n-1}u^m v^p s\rangle - \frac{\alpha n}{2}\langle\tilde{h}^{n-1}u^{m+2}v^p s\rangle - \frac{\alpha n}{2}\langle\tilde{h}^{n-1}u^m v^{p+2} s\rangle \\ & + n(n-1)k_{xx}(0,0)\langle\tilde{h}^{n-2}u^m v^p s\rangle - m(m-1)k_{xx}(0,0)\langle\tilde{h}^n u^{m-2}v^p s\rangle \\ & - p(p-1)k_{xx}(0,0)\langle\tilde{h}^n u^m v^{p-2} s\rangle. \end{aligned} \tag{100}$$

Starting from eqs.(54),(61) and (76) all the $\langle s\tilde{h}^n u^m v^p\rangle$ moments can be evaluated and it can be shown that, with flat initial condition, they are equal to zero. So we conclude $\langle s\Theta\rangle = \langle u_y\Theta\rangle = \langle v_x\Theta\rangle = 0$.

REFERENCES

[1] M. Kardar, G. Parisi and Y.C. Zhang, Phys. Rev. Lett. **56**,889 (1986).

[2] H. Fogedby, cond-mat/0201168 and references therein.

[3] L. Golubovic and Z. G. Wang, Phys. Rev. E**49**,2567 (1994).

[4] I. Ispolatov, P. L. Krapirsky and S. Redner, Phys. Rev. E**52**, 2540 (1995).

[5] J. Krug and H. Spohn, Europhys. Lett. **8**, 219 (1989).

[6] H. Van Beijeren, R. Kutner and H. Spohn, Phys. Rev. Lett. **54**,2026(1985).

[7] H. K. Janseen and B. Schmittmann, Z. Phys. B: Condensed Matter **63**,517(1986).

[8] D. A. Huse, C. L. Henley and D.S. Fisher, Phys. Rev. Lett. **55**, 2924 (1985); M. Kardar and Y.C. Zhang, Phys. Rev. Lett. **58**, 2087 (1987) ; D.S. Fisher and D.A. Huse, Phys. Rev. B **43**, 10728 (1991).

[9] M.V. Feigelman, Sov. Phys, JETP **52** (1980) 555; translated from Zh. Eksper. Teoret. Fiz. **79** (1980), 1095.

[10] J. Krug and H. Spohn in *Solids far from equelibrium growth, morphology and defects*, edited by C. Godreche (Cambridge University Press, New York, 1990).

[11] U. Frisch, J. Bec, B. Villone, Physica D **152-153**, 620 (2001).

[12] S. F. Shandarian and Ya. B. Zeldovich, Rev. Mod. Phys. **61**, 185 (1989).

[13] U. Frisch, J. Bec, nlin. CD/0012033, proceedings Les Houches 2000 "New Trends in Turbulence"; J. Davoudi, A.A. Masoudi, M. Reza Rahimi Tabar, A. Reza Rastegar and F. Shahbazi, Phys. Rev. E **63** 056308 (2001)

[14] A.A. Masoudi, F. Shahbazi, J.Davoudi and M.R. Rahimi Tabar, Phys. Rev. **E 65**, 026132 (2002).

[15] V. Arnol'd, Yu. Baaryshnikov and I. Bogaevski, in S. Gurbatov, A. Malakhov and A. Saichev, *Non-linear random waves and turbulence in nondispersive media: waves, rays and particles*(Manchester University press, Manchester, 1991), p. 290.

[16] J. Bec, R. Iturriaga and K. Khanin, nlin. CD/0112050 v1.